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# Spectrum generating functions for non-canonical quantum oscillators

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**Abstract**
The $n$-dimensional (isotropic and non-isotropic) harmonic oscillator is studied as a Wigner quantum system. In particular, we focus on the energy spectrum of such systems. After briefly recalling the notion of a Wigner quantum system, we show how to solve the compatibility conditions in terms of $\mathfrak{osp}(1|2n)$ generators, and also recall the solution in terms of $\mathfrak{gl}(1|n)$ generators. We then go on to describe a general method for determining a spectrum generating function for an element of the Cartan subalgebra when working with a representation of any Lie (super)algebra. Herein, the character of the representation at hand plays a crucial role. This method is then applied to the $n$-dimensional isotropic harmonic oscillator, yielding explicit formulae for the energy eigenvalues and their multiplicities. This is done using various interesting computational results from the field of symmetric and supersymmetric Schur functions.

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## 1. Introduction

Harmonic oscillator models are among the most studied both in classical physics and quantum mechanics, due to the fact that they are analytically solvable and because of their numerous applications [1]. In the quantum approach the position and momentum operators ($\hat{q}$ and $\hat{p}$ respectively) satisfy the canonical commutation relations $[\hat{p}, \hat{q}] = -i\hbar$, and the model is described by its Hamiltonian and the Heisenberg equations (in the Heisenberg picture). The spectrum of the Hamiltonian is of paramount importance as it yields the values that might come up when measuring the total energy of the system.

Already in 1950 Wigner asked himself the question whether the canonical commutation relations (CCRs) determine the equations of motion [2]. In that same paper he answered this question by showing that requiring the compatibility between the Heisenberg and Hamilton equations does not imply the CCRs between position and momentum operators. He showed that there are (self-adjoint) operators $\hat{p}$ and $\hat{q}$ for which the Hamilton and Heisenberg equations are equivalent (as operator equations) but for which it no longer holds that $[\hat{p}, \hat{q}] = -\mathrm{i}\hbar$. This very fundamental generalization of the quantum harmonic oscillator is nowadays known as the 'Wigner quantum oscillator'. The deformation is characterized by a positive parameter $a$ and the CCRs are satisfied only when $a = 1/2$. It is now known that this parameter $a$ can in fact be viewed as the parameter characterizing a unitary irreducible representation of the Lie superalgebra $\mathfrak{osp}(1|2)$ [3]. This Wigner quantum oscillator is an example of a 'Wigner quantum system' (WQS). Such systems, introduced much later by Palev [3–5], refer to a quantum mechanical system described by a Hamiltonian $\hat{H}$ (as a function of position and momentum operators), for which the CCRs are not imposed, but instead for which the equivalence of the Heisenberg equations and Hamilton's equations is postulated (referred to as the compatibility conditions).

WQSs belong to the field of non-standard quantization, or more precisely to the class of models of non-commutative quantum systems. Nowadays there is quite some interest in such models, or more generally in theories with an underlying non-commutative geometry [6–10]. The interest is not only purely theoretical, but also inspired e.g. by the prediction of string theory that the geometry of space becomes non-commutative at very small distances [11]. In this context, a WQS has the advantage that deformations of commutation relations are not put in 'by hand', by inserting some extra deformation parameter. In contrast, in a WQS the non-commutativity (or deformation of the CCRs) simply follows from some other first principles, namely the earlier mentioned compatibility conditions.

Among the quantum systems that have been studied as a WQS, we mention [12–15]. Most attention went to multi-dimensional (isotropic) oscillators as WQS [5, 16–19], and to linear chains of one-dimensional harmonic oscillators coupled by a nearest neighbour interaction [20, 21]. In the last example, a solution to the so-called compatibility conditions, expressing the equivalence of the Hamilton and the Heisenberg equations, was given in terms of the Lie superalgebra $\mathfrak{gl}(1|n)$, for which unitary irreducible representations are known.

Quite recently, the paraboson Fock space for the Lie superalgebra $\mathfrak{osp}(1|2n)$ was constructed [22]. This is a lowest weight representation characterized by a positive parameter $p$ (subject to some conditions) and we will denote these representations as $V(p)$. In [22], an explicit basis for the representation space is given, along with the matrix elements of the representation. The characters and some character formulae are also given. The construction of this representation made it worthwhile to go and look for solutions of the compatibility conditions of quantum systems in terms of generators of the Lie superalgebra $\mathfrak{osp}(1|2n)$, which is the main topic of this paper.

In this paper, we reconsider the isotropic and non-isotropic $n$-dimensional quantum harmonic oscillator as a WQS. In section 2, we briefly review the fundamentals of WQSs and explain how the compatibility conditions for the current system are derived. We then give a new solution for the non-isotropic oscillator in terms of the odd generators of the Lie superalgebra $\mathfrak{osp}(1|2n)$ (or equivalently in terms on $n$ pairs of paraboson operators). Apart from this, there is a second solution of these compatibility conditions in terms of the Lie superalgebra $\mathfrak{gl}(1|n)$. This result is in fact an easy consequence of the results in [20].

Since Lie superalgebras on themselves do not give a suitable framework for studying the behaviour of the operators in a WQS, one has to work with (unitary irreducible) representations of these algebras. In section 3, we present the representations that are going to be used in this paper. In fact, we will only describe their characters (and some character formulae) as this is all that is needed in order to derive the spectrum generating function. The characters of the representations are expressed in terms of (supersymmetric) Schur functions.

In section 4, we consider the quite general problem of determining the spectrum of any element of the Cartan subalgebra in a representation of a Lie (super)algebra. A spectrum generating function, i.e. a formal power series in some variable, where the exponents give the eigenvalues and the coefficients the multiplicity of the corresponding eigenvalue, is easily obtained by performing a simple substitution in the character of the representation.

In the next section, we apply this technique to the $\mathfrak{osp}(1|2n)$ solution of the compatibility conditions of the $n$-dimensional isotropic harmonic oscillator (with frequency $\omega$). We immediately obtain that for any admissible value of $p > 0$ the representation $V(p)$ yields a countable infinite and equidistant spectrum with spacing $\hbar\omega$ and ground level $\hbar\omega np/2$. Also, the degeneracies of the energy levels are seen to be polynomials in $n$. We then study the spectrum more thoroughly for some specific values of $p$, and we see for instance that we recover the known results for the canonical case, i.e. when $p = 1$. Furthermore, it is shown that the multiplicities of the eigenvalues in the case when $p \in \{1, 2, \ldots, n-1\}$ (non-generic cases) may be determined in terms of the multiplicities for the case $p > n-1$ (generic case). Finally, the three-dimensional oscillator is considered as an example.

In section 6 the spectrum generating function technique is applied to the irreducible covariant tensor representations of $\mathfrak{gl}(1|n)$. These representations are less like the canonical solution since they are finite dimensional. Nevertheless, also in this case the spectrum is equidistant with spacing $\hbar\omega$, and the multiplicities of the different energy levels can again be seen as polynomials in $n$.

## 2. The quantization procedure and some of its solutions

In this section, we briefly describe how to derive the compatibility conditions for the $n$-dimensional harmonic oscillator and also give some of their solutions in terms of Lie superalgebra generators. To be more specific, consider the Hamiltonian for an $n$-dimensional harmonic oscillator with mass $m$ and frequencies $\omega_j$ $(j = 1, \ldots, n)$:

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{n} \hat{p}_j^2 + \frac{m}{2}\sum_{j=1}^{n} \omega_j^2 \hat{q}_j^2. \tag{2.1}$$

Here, the position and momentum operators are given by $\hat{q}_j$ and $\hat{p}_j$ respectively. We shall treat both the non-isotropic case ($\omega_j$'s different) as the isotropic case (all $\omega_j$'s equal). When treating this system as a WQS, one no longer requires the CCRs between position and momentum operators, but instead one requires the compatibility of the Hamilton and Heisenberg equations. Expressing this compatibility yields the so-called compatibility conditions (CCs).

In this case, the Hamilton equations are

$$\dot{\hat{q}}_j = \frac{\partial \hat{H}}{\partial \hat{p}_j} = \frac{1}{m}\hat{p}_j, \qquad \dot{\hat{p}}_j = -\frac{\partial \hat{H}}{\partial \hat{q}_j} = -m\omega_j^2 \hat{q}_j, \qquad j = 1, \ldots, n, \tag{2.2}$$

while the Heisenberg equations are

$$\dot{\hat{q}}_j = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{q}_j], \qquad \dot{\hat{p}}_j = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{p}_j], \qquad j = 1, \ldots, n. \tag{2.3}$$

The compatibility conditions are thus

$$[\hat{H}, \hat{q}_j] = -\mathrm{i}\frac{\hbar}{m}\hat{p}_j, \qquad [\hat{H}, \hat{p}_j] = \mathrm{i}\hbar m\omega_j^2 \hat{q}_j, \qquad j = 1, \ldots, n. \tag{2.4}$$

One then typically introduces the following linear combinations of the unknown operators $\hat{q}_j$ and $\hat{p}_j$:

$$a_j^{\mp} = \sqrt{\frac{m\omega_j}{2\hbar}}\hat{q}_j \pm \frac{\mathrm{i}}{\sqrt{2m\hbar\omega_j}}\hat{p}_j, \qquad j = 1, \ldots, n. \tag{2.5}$$

The Hamiltonian has the following easy expression in terms of the operators $a_j^{\pm}$:

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left(a_j^+a_j^- + a_j^-a_j^+\right) = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left\{a_j^+, a_j^-\right\}. \tag{2.6}$$

It is now easy to verify that the compatibility conditions (2.4) are equivalent with

$$\left[\sum_{j=1}^{n}\omega_j\left\{a_j^+, a_j^-\right\}, a_k^{\pm}\right] = \pm 2\omega_k a_k^{\pm}, \qquad k = 1, \ldots, n. \tag{2.7}$$

Finally, due to the fact that the position and momentum operators are self-adjoint, one has that

$$\left(a_j^{\pm}\right)^{\dagger} = a_j^{\mp}, \qquad j = 1, \ldots, n. \tag{2.8}$$

So in conclusion, solving the compatibility conditions amounts to finding operators $a_j^{\pm}$ $(j = 1, \ldots, n)$, acting in some Hilbert space, that satisfy equations (2.7), subject to (2.8). Note that the $a_j^{\pm}$, in general, do not satisfy the usual boson commutation relations, since the CCR's are not required.

In a sense, (2.7) can be considered as a generalization of the boson commutation relations. These are now 'triple commutation relations', which are automatically satisfied for ordinary boson operators. More general solutions of (2.7) can be found by means of Lie superalgebra generators.

*The* $\mathfrak{osp}(1|2n)$ *solution.* One class of solutions follows by identifying the operators $a_j^{\pm}$ with generators of the Lie superalgebra $\mathfrak{osp}(1|2n)$, or equivalently, with paraboson operators. Indeed, consider a system consisting of $n$ pairs of paraboson operators whose defining relations are given by

$$\left[\left\{b_j^{\xi}, b_k^{\eta}\right\}, b_l^{\epsilon}\right] = (\epsilon - \xi)\delta_{jl}b_k^{\eta} + (\epsilon - \eta)\delta_{kl}b_j^{\xi}, \tag{2.9}$$

where $j, k, l \in \{1, 2, \ldots, n\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and $-1$ in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$). It is known that the Lie superalgebra generated by the odd elements $b_j^{\pm}$ $(j = 1, \ldots, n)$ subject to the relations (2.9) is in fact the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [23].

Using the triple relations (2.9), it is an easy verification that

$$a_j^- = \sigma_j b_j^-, \qquad a_j^+ = \sigma_j^* b_j^+, \qquad \text{with} \qquad |\sigma_j|^2 = 1 \qquad j = 1, \ldots, n, \tag{2.10}$$

indeed satisfies (2.7). Furthermore, the conditions $\left(a_j^{\pm}\right)^{\dagger} = a_j^{\mp}$ lead to the relations $\left(b_j^{\pm}\right)^{\dagger} = b_j^{\mp}$ for the paraboson operators. So in the following, we shall work with $\mathfrak{osp}(1|2n)$ representations in which these conditions are automatically satisfied, namely the so-called paraboson Fock spaces $V(p)$.

Note that the algebraic form of the Hamiltonian is as follows:

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left\{a_j^-, a_j^+\right\} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left\{b_j^-, b_j^+\right\} = \hbar\sum_{j=1}^{n}\omega_j h_j, \tag{2.11}$$

where the operators $h_j = \{b_j^-, b_j^+\}/2$ $(j = 1, \dots, n)$ span the Cartan subalgebra of $\mathfrak{osp}(1|2n)$ (see [22] for a definition of $\mathfrak{osp}(1|2n)$ in terms of the paraboson operators $b_j^{\pm}$).

*The* $\mathfrak{gl}(1|n)$ *solution.* A second class of solutions for (2.7) can be given in terms of the Lie superalgebra $\mathfrak{gl}(1|n)$. In fact, equations which are equivalent to (2.7) have been encountered before when treating a linear chain of coupled harmonic oscillators as a WQS [20]. After introducing 'normal coordinates', the form of the Hamiltonian of such a chain is essentially given by (2.1). This means that the solution of the CCs found in [20] carries over to this case. We thus have the following solution in terms of odd $\mathfrak{gl}(1|n)$ generators:

$$a_j^- = \sqrt{\frac{2|\beta_j|}{\omega_j}}e_{j0}, \qquad a_j^+ = \operatorname{sign}(\beta_j)\sqrt{\frac{2|\beta_j|}{\omega_j}}e_{0j}, \qquad j = 1, \dots, n, \tag{2.12}$$

with

$$\beta_j = -\omega_j + \frac{1}{n-1}\sum_{k=1}^{n}\omega_k, \qquad j = 1, \dots, n. \tag{2.13}$$

The frequencies are supposed to be such that the constants $\beta_j$ $(j = 1, \dots, n)$ are non-zero. Also, the $e_{0j}$ and $e_{j0}$ are the odd generators of the Lie superalgebra $\mathfrak{gl}(1|n)$. The fact that (2.12) is indeed a solution of the CCs is easily checked using the commutation and anti-commutation relations in $\mathfrak{gl}(1|n)$:

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})}\delta_{il}e_{kj}. \tag{2.14}$$

The elements $e_{0j}$ and $e_{j0}$ $(j = 1, \dots, n)$ are the odd elements and hence have degree 1. All other basis elements are even elements and have degree 0.

We are going to use the following 'star-condition' for $\mathfrak{gl}(1|n)$ (corresponding to the real form $\mathfrak{u}(1|n)$):

$$(e_{0j})^\dagger = e_{j0}.$$

This is equivalent to the conditions (2.8) for the operators $a_j^{\pm}$ provided all constants $\beta_j$ are positive. Thus in the rest of this paper, we shall assume that the frequencies $\omega_j$ are such that all $\beta_j > 0$ $(j = 1, \dots, n)$, at least when we are working with the $\mathfrak{gl}(1|n)$ solution. The unitary irreducible representations of $\mathfrak{gl}(1|n)$ then yield the appropriate spaces for our operators to act in. On an algebraic level, it is easily verified that the Hamiltonian (2.6) is given by

$$\hat{H} = \hbar\left(\beta e_{00} + \sum_{j=1}^{n}\beta_j e_{jj}\right),$$

with $\beta = \sum_{j=1}^{n}\beta_j$ and $\beta_j$ given by (2.13). Note that the Hamiltonian is again an element of the Cartan subalgebra of $\mathfrak{gl}(1|n)$.

## 3. Some representations of $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ and their characters

Since our goal is to study the spectrum of the Hamiltonian when acting in some Hilbert space, we now introduce the representations we are going to use and in which the Hamiltonian acts. This amounts to collecting some material on a class of representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$, and of the Lie superalgebra $\mathfrak{gl}(1|n)$.

*A class of* $\mathfrak{osp}(1|2n)$ *representations.* For the $\mathfrak{osp}(1|2n)$ solution, we are going to work with the paraboson Fock space $V(p)$, which is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$, with lowest weight $(p/2, p/2, \dots, p/2)$. The parameter $p$, which is sometimes called the order of the paraboson system and which characterizes the representation, is subject to certain constraints. In [22], an explicit basis and the matrix elements of this representation were constructed and also the weight structure (characters and character formulae) was given. Recall also that for $p = 1$ the paraboson Fock space $V(1)$ coincides with the ordinary boson Fock space (so in that case the CCRs are satisfied). When $p \neq 1$, we are dealing with 'deformations' of the CCRs. The main result concerning the unitarity and weight structure of the representations $V(p)$ is the following ([22], theorem 7):

**Theorem 1.** *The* $\mathfrak{osp}(1|2n)$ *representation* $V(p)$ *with lowest weight* $\left(\frac{p}{2}, \ldots, \frac{p}{2}\right)$ *is a unitary irreducible representation if and only if* $p \in \{1, 2, \ldots, n-1\}$ *or* $p > n-1$.

*For* $p > n-1$, *one has*

$$\mathrm{char}\, V(p) = \frac{(x_1 \cdots x_n)^{p/2}}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k)} \tag{3.1}$$

$$= (x_1 \cdots x_n)^{p/2} \sum_\lambda s_\lambda(x). \tag{3.2}$$

*For* $p \in \{1, 2, \ldots, n-1\}$, *the character of* $V(p)$ *is given by*

$$\mathrm{char}\, V(p) = (x_1 \cdots x_n)^{p/2} \sum_{\lambda, \ell(\lambda) \leqslant p} s_\lambda(x) \tag{3.3}$$

*where* $\ell(\lambda)$ *is the length of the partition* $\lambda$.

In this theorem, $s_\lambda(x)$ stands for the Schur symmetric function [24], and although in (3.2) no restriction on the length of the partitions is given, the sum is in effect over all partitions of length at most $n$ since $s_\lambda(x_1, \ldots, x_n)$ vanishes if $\ell(\lambda) > n$. The length of a partition is its number of parts. This and all other notions involving partitions may for instance be found in [24].

For our purposes, it is interesting to note that in the case when $p \in \{1, 2, \ldots, n-1\}$, the character of the representation $V(p)$ can also be written as follows [22]:

$$\mathrm{char}\, V(p) = (x_1 \cdots x_n)^{p/2} \frac{\mathbf{E}_{(0,p)}}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k)}, \tag{3.4}$$

with

$$\mathbf{E}_{(0,p)} = \sum_\eta (-1)^{c_\eta} s_\eta(x_1, \ldots, x_n), \tag{3.5}$$

where the sum is over all partitions $\eta$ of the form

$$\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 + p & a_2 + p & \cdots & a_r + p \end{pmatrix} \equiv (a_1, a_2, \ldots, a_r | a_1 + p, a_2 + p, \ldots, a_r + p) \tag{3.6}$$

in Frobenius notation, and

$$c_\eta = a_1 + a_2 + \cdots + a_r + r. \tag{3.7}$$

The Frobenius notation is a special way of denoting partitions [24] related to the lengths of the rows and columns in the Young diagram of the partition, counted from the diagonal. In the current case, the partitions $\eta$ are all those with a Young diagram of shape given in figure 1. The number of terms in the numerator of (3.4) is $2^{n-p}$. This follows immediately from the fact that $s_\lambda(x_1, \ldots, x_n)$ vanishes identically if $\ell(\lambda) > n$. The length of a partition $\eta$ given by (3.6) is given by $1 + a_1 + p$, hence $a_1 \leqslant n - p - 1$. It is easily checked by induction

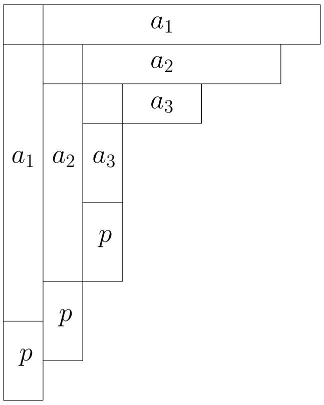



**Figure 1.** Typical shape of the Young diagram for the partition $\eta$, given by the Frobenius notation (3.6) (illustrated here for $r = 3$).

that the number of partitions of the form $\eta$ with $a_1 \leqslant k$ is given by $2^{k+1}$, since it is immediately clear that the $a_i$ are a strictly decreasing sequence of non-negative integers.

Some interesting special cases of (3.5) are

$$\mathbf{E}_{(0,1)} = \prod_{1\leqslant j<k\leqslant n} (1 - x_j x_k), \tag{3.8}$$

and

$$\mathbf{E}_{(0,n-1)} = 1 - x_1 x_2 \cdots x_n. \tag{3.9}$$

These lead to

$$\operatorname{char} V(1) = (x_1 \cdots x_n)^{1/2} \frac{1}{\prod_i (1 - x_i)}, \tag{3.10}$$

and

$$\operatorname{char} V(n-1) = (x_1 \cdots x_n)^{(n-1)/2} \frac{(1 - x_1 x_2 \cdots x_n)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k)}. \tag{3.11}$$

*A class of* $\mathfrak{gl}(1|n)$ *representations.* We now turn to the characters of the $\mathfrak{gl}(1|n)$ representations we are going to use. It is well known that the symmetric Schur functions are the characters of the irreducible covariant tensor representations of $\mathfrak{gl}(m)$. Berele and Regev showed that the characters of irreducible covariant tensor representations of $\mathfrak{gl}(m|n)$ are supersymmetric Schur functions [25].

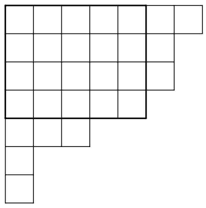
Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be two sets of independent variables. The complete supersymmetric functions $h_r(x|y)$, with $r$ a non-negative integer, are defined as

$$h_r(x|y) = \sum_{k=0}^{r} h_{r-k}(x) e_k(y), \tag{3.12}$$

where $h$ and $e$ on the right-hand side denote the ordinary complete and elementary symmetric polynomials respectively. One then has the following determinantal formula for supersymmetric Schur polynomials indexed by a partition $\lambda$:

$$s_\lambda(x|y) = \det(h_{\lambda_i - i + j}(x|y))_{1\leqslant i,j\leqslant \ell(\lambda)}. \tag{3.13}$$

$F^\lambda =$



with

$$\lambda = ((n^m) + \tau) \cup \eta$$
$$\tau = (2, 1, 1)$$
$$\eta = (3, 1, 1)$$

**Figure 2.** Example of partition for which the supersymmetric Schur polynomial will factorize. In this case, we assume that $m = 4$ and $n = 5$. Note that $\lambda_m = 5 \geqslant n = 5 \geqslant \lambda_{m+1} = 3$.

Formula (3.13) is the analogue of the Jacobi–Trudi formula for symmetric Schur functions. However, there do exist a number of other formulae for the supersymmetric Schur polynomials. One is a combinatorial formula in terms of supertableaux (just as there is a formula for Schur polynomials in terms of tableaux), and from this combinatorial formula, one deduces the following expansion of supersymmetric Schur polynomials in terms of ordinary Schur polynomials ([24], section I.5, example 23):

$$s_\lambda(x|y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_{\nu'}(y), \tag{3.14}$$

where the coefficients $c^\lambda_{\mu\nu}$ are the Littlewood–Richardson coefficients [24, 26] (the coefficients in the expansion of a product of two Schur functions as a linear combination of Schur functions). They are non-negative integers and may be determined by a combinatorial rule, the so-called Littlewood–Richardson rule. In (3.14), $\nu'$ denotes the conjugate partition of $\nu$, i.e. the partition whose Young diagram is the transpose of that of $\nu$.

The polynomials $s_\lambda(x|y)$ are identically zero when $\lambda_{m+1} > n$, so we are considering only those partitions for which $\lambda_{m+1} \leqslant n$. Supersymmetric Schur polynomials $s_\lambda(x|y)$ indexed by such a partition are the characters of the irreducible covariant tensor representations of $\mathfrak{gl}(m|n)$ [25].

In the case when $\lambda_m \geqslant n$, there exists a particular convenient formula that expresses a supersymmetric Schur polynomial as a product of two ordinary Schur symmetric polynomials multiplied by the variables associated with the $(m, n)$ rectangle in the upper left corner of the Young diagram [27, 28]. Indeed, let $\lambda = ((n^m) + \tau) \cup \eta$, then

$$s_\lambda(x_1, \ldots, x_m | y_1, \ldots, y_n) = s_\tau(x_1, \ldots, x_m) s_{\eta'}(y_1, \ldots, y_n) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j). \tag{3.15}$$

In figure 2, this is illustrated for $m = 4$, $n = 5$.

## 4. Spectrum generating functions for Cartan subalgebra elements

Suppose we work with a Lie (super)algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, and let the Cartan subalgebra be spanned by the elements $h_j$ $(j = 1, \ldots, n)$. For some given constants $\alpha_j$, we consider the element

$$C = \sum_{j=1}^{n} \alpha_j h_j, \tag{4.1}$$

and our aim is to determine the spectrum (including degeneracies) of such elements $C$ when working in a particular representation $R$ of $\mathfrak{g}$. The representation is supposed to be unitary and irreducible. We also assume that the character of the representation is known:

$$\text{char}\, R = \sum_{\mathbf{r}} d_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} = \sum_{r_1,\ldots,r_n} d_{r_1,\ldots,r_n} x_1^{r_1} \cdots x_n^{r_n}.$$

The character is a formal power series consisting of terms $d_{r_1,\ldots,r_n} x_1^{r_1} \cdots x_n^{r_n}$ with $(r_1, \ldots, r_n)$ a weight of the representation and with $d_{r_1,\ldots,r_n}$ the dimension of the corresponding weight space. A method to turn the character of such a representation into a spectrum generating function for $C$ is common knowledge, but is not so easy to trace in the literature. So we briefly outline it in this section.

Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a multi-index (weight) such that $d_{\mathbf{r}} \neq 0$, then there exist $d_{\mathbf{r}}$ linearly independent basis vectors $|m\rangle$ of the representation for which $h_j|m\rangle = r_j|m\rangle$ $(j = 1, \ldots, n)$ and hence

$$C|m\rangle = \left(\sum_{j=1}^{n} \alpha_j h_j\right)|m\rangle = \left(\sum_{j=1}^{n} \alpha_j r_j\right)|m\rangle = C_{\mathbf{r}}|m\rangle.$$

The eigenvalues of $C$ (not necessarily all different) are thus given by

$$C_{\mathbf{r}} = \sum_{j=1}^{n} \alpha_j r_j$$

for each weight $\mathbf{r} = (r_1, \ldots, r_n)$. Let $t$ be a new variable. If one performs the substitution

$$x_j \to t^{\alpha_j}, \qquad (j = 1, \ldots, n) \tag{4.2}$$

in the character, then one gets

$$\text{spec}\, C = \sum_{r_1,\ldots,r_n} d_{r_1,\ldots,r_n} t^{\alpha_1 r_1} \cdots t^{\alpha_n r_n} = \sum_{\mathbf{r}} d_{\mathbf{r}} t^{C_{\mathbf{r}}}. \tag{4.3}$$

Clearly, the different eigenvalues of $C$ in this particular representation are read of as the exponents of $t$. If all eigenvalues $C_{\mathbf{r}}$ are different, their multiplicities are given by the coefficients $d_{\mathbf{r}}$ in the character. It might happen, however, that not all eigenvalues $C_{\mathbf{r}}$ are different. In that case, one should collect equal powers of $t$ in spec $C$, and the coefficient of $t^{C_{\mathbf{r}}}$ then gives the multiplicity of this particular eigenvalue.

To summarize, in order to find the spectrum of an element $C$ of the form (4.1) in the representation $R$, one has to perform the substitutions (4.2) in the character of $R$. The eigenvalues are then read of as the exponents (of $t$), while their degeneracies are given by the corresponding coefficients. So spec $C$, given by (4.3), is a spectrum generating function for the operator $C$ in the representation $R$.

## 5. Spectrum of the Hamiltonian in the $\mathfrak{osp}(1|2n)$ solution

The purpose of this section is to study the spectrum of $\hat{H}$ in the unitary representations $V(p)$ of $\mathfrak{osp}(1|2n)$, and more particularly to determine the spectrum generating function for $\hat{H}$. In particular, for $p = 1$, we should find back the spectrum of the canonical quantum oscillator.

Following the technique of the previous section, constructing the spectrum generating function is almost straightforward. The most interesting case is the isotropic oscillator, which will get further attention.

*Algebraic form of the Hamiltonian.* As mentioned in section 2, the Hamiltonian of the system is given by

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left\{a_j^-, a_j^+\right\} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\left\{b_j^-, b_j^+\right\} = \hbar\sum_{j=1}^{n}\omega_j h_j, \tag{5.1}$$

where the operators $h_j = \left\{b_j^-, b_j^+\right\}/2$ $(j = 1, \ldots, n)$ span the Cartan subalgebra of $\mathfrak{osp}(1|2n)$. In this section, we shall pay particular attention to the isotropic case, with $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$. In that case, the expression for the spectrum generating function simplifies a lot.

*Energy levels in terms of Schur polynomials.* In general, since all constants $\alpha_j$ in (4.1) are identical for the isotropic oscillator, one has to perform the following substitutions in the character of $V(p)$:

$$x_j \to t^{\hbar\omega} =: z, \qquad (j = 1, \ldots, n) \tag{5.2}$$

in order to obtain the spectrum generating function for $\hat{H}$. This spectrum generating function will be denoted by spec $\hat{H}$. Whether $p \in \{1, 2, \ldots, n-1\}$ or $p > n-1$, the character can always be written as

$$\text{char}\, V(p) = (x_1 \cdots x_n)^{p/2} \sum_{\lambda, \ell(\lambda)\leqslant\lceil p\rceil} s_\lambda(x_1, \ldots, x_n). \tag{5.3}$$

(Note: the ceiling function is only necessary to take care of the cases $n-1 < p < n$.) After performing the substitutions (5.2), we will have specialized the Schur polynomials to

$$s_\lambda(z, \ldots, z) = z^{|\lambda|} s_\lambda(1, \ldots, 1).$$

Here, we have used the fact that the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is homogeneous of degree $|\lambda|$. There is a known expression for such specializations $s_\lambda(1, \ldots, 1)$ of Schur polynomials in terms of the contents and the hook lengths of the defining partition $\lambda$ ([24], I.3, example 4):

$$s_\lambda(1, \ldots, 1) = \prod_{(i,j)\in\lambda} \frac{n + c(i, j)}{h(i, j)}, \tag{5.4}$$

where $c(i, j) = j - i$ and $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$ is the content and the hook length of $(i, j)$ respectively. Alternatively, (5.4) is also the dimension of the $\mathfrak{gl}(n)$ irreducible representation labelled by the partition $\lambda$. Following Macdonald [24], we introduce the following generalization of the binomial coefficients for any partition $\lambda$:

$$\binom{X}{\lambda} = \prod_{(i,j)\in\lambda} \frac{X - c(i, j)}{h(i, j)}. \tag{5.5}$$

On the left of figure 3 the numerator of each factor in (5.5) is shown for a certain partition, while on the right of that same figure the denominators (hook lengths) associated with each block are shown.

For the specializations of the Schur functions one thus has

$$s_\lambda(1, \ldots, 1) = \prod_{(i,j)\in\lambda} \frac{n + c(i, j)}{h(i, j)} = \prod_{(i,j)\in\lambda} \frac{n - c(j, i)}{h(j, i)} = \binom{n}{\lambda'}. \tag{5.6}$$

| | | | |
|---|---|---|---|
| $X$ | $X-1$ | $X-2$ | $X-3$ |
| $X+1$ | $X$ | $X-1$ | |
| $X+2$ | $X+1$ | $X$ | |
| $X+3$ | $X+2$ | | |

| | | | |
|---|---|---|---|
| 7 | 6 | 4 | 1 |
| 5 | 4 | 2 | |
| 4 | 3 | 1 | |
| 2 | 1 | | |

**Figure 3.** Illustration of the various factors of a generalized binomial coefficient.

It is easily checked that for $\lambda = (k)$,

$$\binom{X}{(k)} = \frac{X}{k}\frac{X-1}{k-1}\cdots\frac{X-k+1}{1} = \frac{(X-k+1)_k}{k!} = \binom{X}{k},$$

so that for each partition of length 1, the generalized binomial coefficient becomes an ordinary binomial coefficient. The notation $(a)_k$ stands for the rising factorial or Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$ if $k > 0$ and $(a)_0 = 1$.

Putting all of this together, we get for the spectrum generating function for the isotropic oscillator:

$$\text{spec}\,\hat{H} = z^{np/2}\sum_{\lambda,\ell(\lambda)\leqslant\lceil p\rceil} s_\lambda(z,\ldots,z) = \sum_{k\geqslant 0}\sum_{\lambda,|\lambda|=k,\ell(\lambda)\leqslant\lceil p\rceil} s_\lambda(1,\ldots,1)t^{\hbar\omega(np/2+k)}. \tag{5.7}$$

From this, it is clear that we will have equidistant energy levels

$$E_k^{(p)} = \hbar\omega(np/2+k), \qquad k = 0,1,2,3,\ldots \tag{5.8}$$

with spacing $\hbar\omega$ and with multiplicities (degeneracies)

$$\mu\big(E_k^{(p)}\big) \equiv \sum_{\lambda,|\lambda|=k,\ell(\lambda)\leqslant\lceil p\rceil} s_\lambda(1,\ldots,1) = \sum_{\lambda,|\lambda|=k,\ell(\lambda)\leqslant\lceil p\rceil}\binom{n}{\lambda'}. \tag{5.9}$$

From definition (5.5) it is clear that the generalized binomial coefficient $\binom{X}{\lambda}$ is a polynomial of degree $|\lambda|$ in the variable $X$. This means that in general, the degeneracy (or multiplicity) $\mu\big(E_k^{(p)}\big)$ of the $k$th energy level $E_k^{(p)}$ is a polynomial of degree $k$ in $n$. (Clearly the degree is going to be at most $k$, and since the coefficient of $n^{|\lambda|}$ in (5.6) is positive, the degree is exactly $k$.) Since the degeneracy $\mu\big(E_k^{(p)}\big)$ is in fact independent of $p$ in the generic case, i.e. when $p > n-1$, we will drop the superscript $(p)$ in this case.

*The canonical solution* $(p = 1)$. The representation $V(1)$ of $\mathfrak{osp}(1|2n)$ is nothing but the canonical solution of the harmonic oscillator model, i.e. the CCRs are satisfied in this case. Application of the described technique should thus give the known result for the spectrum of the Hamiltonian. In this case, there is a very simple expression for char $V(1)$, namely (3.10). Performing the substitutions (5.2) immediately yields

$$\text{spec}\,\hat{H} = \frac{z^{n/2}}{(1-z)^n} = z^{n/2}\sum_{k\geqslant 0}\binom{n+k-1}{k}z^k = \sum_{k\geqslant 0}\binom{n+k-1}{k}t^{\hbar\omega(n/2+k)}.$$

We thus see that we indeed have equidistant energy levels, with spacing $\hbar\omega$, and that the ground energy level is given by $\hbar\omega n/2$. The multiplicity of the $k$th energy level is given by the binomial coefficient $\binom{n+k-1}{k}$, which is, as it should be, a polynomial of degree $k$ in the variable $n$. These results are of course not new, but they do coincide with known results [29–31].

The same result is also easily obtained from (5.9):

$$\mu\left(E_k^{(1)}\right) = \binom{n}{(k)'} = (-1)^k \binom{-n}{(k)} = (-1)^k \binom{-n}{k} = \binom{n+k-1}{k}, \tag{5.10}$$

where the following property of generalized binomial coefficients was used

$$\binom{X}{\lambda} = (-1)^{|\lambda|} \binom{-X}{\lambda'},$$

together with the fact that for a partition of length 1, the generalized binomial coefficient coincides with the classical binomial coefficient.

*The case* $p = 2$. Also when $p = 2$, i.e. for the representation $V(2)$, the multiplicities of each energy level can be determined explicitly even though there is no 'closed form' character formula for this representation. Consider a partition $(\lambda_1, \lambda_2)$ of length (at most) 2. We are now going to use formula (5.6) to find a formula for $s_{(\lambda_1,\lambda_2)}(1, \ldots, 1)$. This is easy, we just have to consider three parts in the Young diagram of the partition: the first $\lambda_2$ blocks on the first row, the last $\lambda_1 - \lambda_2$ blocks on the first row, and the blocks on the second row. For each of these three sets of blocks, it is easy to write down the contents on the hook lengths of the blocks in it. Doing this and multiplying the result yields the following:

$$\begin{aligned} s_{(\lambda_1,\lambda_2)}(1, \ldots, 1) &= \left(\prod_{l=1}^{\lambda_2} \frac{n+l-1}{\lambda_1+2-l}\right) \left(\prod_{l=1}^{\lambda_1-\lambda_2} \frac{n+\lambda_1-l}{l}\right) \left(\prod_{l=1}^{\lambda_2} \frac{n+l-2}{\lambda_2+1-l}\right) \\ &= \frac{(n)_{\lambda_2-1}(n-1)_{\lambda_1+1}(\lambda_1-\lambda_2+1)}{\lambda_2!(\lambda_1+1)!}. \end{aligned}$$

It is interesting to note that this result is also valid when $\lambda_2 = 0$ or even when $\lambda$ is the empty partition. It thus follows from (5.9) that

$$\begin{aligned} \mu\left(E_k^{(2)}\right) &= \sum_{\lambda, |\lambda|=k, \ell(\lambda)\leqslant 2} s_\lambda(1, \ldots, 1) = \sum_{\lambda_1+\lambda_2=k,\, \lambda_1\geqslant\lambda_2\geqslant 0} s_{(\lambda_1,\lambda_2)}(1, \ldots, 1) \\ &= \sum_{\lambda_1=\lceil\frac{k}{2}\rceil}^{k} \frac{(n)_{k-\lambda_1-1}(n-1)_{\lambda_1+1}(2\lambda_1-k+1)}{(k-\lambda_1)!(\lambda_1+1)!} \\ &= \sum_{\lambda_1=\lceil\frac{k}{2}\rceil}^{k} f(\lambda_1). \end{aligned}$$

Now, $f(\lambda_1)$ is a hypergeometric term that is Gosper-summable [32], and indeed it is easy to verify that

$$f(\lambda_1) = g(\lambda_1+1) - g(\lambda_1),$$

with

$$g(\lambda_1) = -\frac{(n)_{k-\lambda_1}(n)_{\lambda_1}}{(k-\lambda_1)!\lambda_1!}.$$

The summation over $f$ thus telescopes, and the multiplicity $\mu\left(E_k^{(2)}\right)$ is given by

$$\mu\left(E_k^{(2)}\right) = g(k+1) - g\left(\left\lceil\frac{k}{2}\right\rceil\right) = \frac{(n)_{k-\lceil\frac{k}{2}\rceil}(n)_{\lceil\frac{k}{2}\rceil}}{\left(k-\lceil\frac{k}{2}\rceil\right)!\left(\lceil\frac{k}{2}\rceil\right)!}, \tag{5.11}$$

since $g(k+1) = 0$. If we consider the even and odd cases separately, the expression (5.11) simplifies even further:

$$\mu\left(E_{2k}^{(2)}\right) = \frac{(n)_k^2}{k!^2}, \qquad \text{and} \qquad \mu\left(E_{2k+1}^{(2)}\right) = \frac{(n)_k^2(n+k)}{k!^2(k+1)}. \tag{5.12}$$

This can also be written as in a form similar to the canonical case (5.10):

$$\mu\left(E_{2k}^{(2)}\right) = \binom{n+k-1}{k}^2, \qquad \text{and} \qquad \mu\left(E_{2k+1}^{(2)}\right) = \binom{n+k-1}{k}\binom{n+k}{k+1}. \tag{5.13}$$

*The generic case.* We now turn our attention to the case where the representations $V(p)$ are generic, i.e. $p > n-1$. In this case, a character formula is given by (3.1). After performing the substitution (5.2), one gets as a spectrum generating function:

$$\begin{aligned}
\operatorname{spec} \hat{H} &= \frac{z^{np/2}}{(1-z)^n(1-z^2)^{\binom{n}{2}}} = \frac{z^{np/2}}{(1-z)^{\binom{n+1}{2}}(1+z)^{\binom{n}{2}}} \\
&= z^{np/2} \sum_{k_1 \geqslant 0} \binom{\binom{n+1}{2}+k_1-1}{k_1} z^{k_1} \sum_{k_2 \geqslant 0} \binom{\binom{n}{2}+k_2-1}{k_2} (-z)^{k_2} \\
&= \sum_{k \geqslant 0} \sum_{l=0}^{k} (-1)^l \binom{\binom{n+1}{2}+k-l-1}{k-l} \binom{\binom{n}{2}+l-1}{l} t^{\hbar\omega(np/2+k)}.
\end{aligned}$$

So, this again confirms that we have equidistant energy levels, with the ground energy level given by $\hbar\omega np/2$, and with spacing $\hbar\omega$. The multiplicity of the $k$th energy level is thus given by

$$\begin{aligned}
\mu(E_k) &= \sum_{l=0}^{k} (-1)^l \binom{\binom{n+1}{2}+k-l-1}{k-l} \binom{\binom{n}{2}+l-1}{l} \\
&= \binom{\binom{n+1}{2}+k-1}{k} {}_2F_1\left(\begin{matrix} -k, \binom{n}{2} \\ 1-k-\binom{n+1}{2} \end{matrix}; -1\right)
\end{aligned} \tag{5.14}$$

where the summation was also written using a Gauss hypergeometric function. For $n = 1$, it follows that $\mu(E_k) = 1$, so one recovers the result of Wigner [2] who observed that the energy levels for the one-dimensional oscillator only shift when using non-canonical solutions, but remain non-degenerate [2]. Note that it is *not* immediately clear that (5.14) in fact defines a polynomial of degree $k$ in $n$.

Putting together (5.14) and (5.9), we can now express a sum of generalized binomial coefficients as a sum of a product of two ordinary binomial coefficients:

$$\sum_{\lambda, |\lambda|=k} \binom{X}{\lambda} = \sum_{l=0}^{k} (-1)^l \binom{\binom{X+1}{2}+k-l-1}{k-l} \binom{\binom{X}{2}+l-1}{l}. \tag{5.15}$$

*The case $p = n-1$.* Another case that can be done explicitly is the case $p = n-1$. A character formula is then given by (3.11) and the spectrum generating function for the isotropic oscillator becomes

$$\begin{aligned}
\text{spec } \hat{H} &= z^{n(n-1)/2} \frac{1 - z^n}{(1-z)^n (1-z^2)^{\binom{n}{2}}} \\
&= z^{n(n-1)/2} (1 - z^n) \sum_{k \geqslant 0} \mu(E_k) z^k \\
&= z^{n(n-1)/2} \left( \sum_{k \geqslant 0} \mu(E_k) z^k - \sum_{k \geqslant n} \mu(E_{k-n}) z^k \right).
\end{aligned}$$

Hence, it is clear that the following holds for the multiplicities $\mu\big(E_k^{(n-1)}\big)$:

$$\mu(E_k^{(n-1)}) = \begin{cases} \mu(E_k) & \text{for} \quad k < n, \\ \mu(E_k) - \mu(E_{k-n}) & \text{for} \quad k \geqslant n. \end{cases} \tag{5.16}$$

*The case* $p \in \{1, 2, \ldots, n-1\}$. For $p \in \{1, 2, \ldots, n-1\}$, the character of $V(p)$ can also be written in the alternative form (3.4). The spectrum generating function is thus (we again use the substitution (5.2)):

$$\begin{aligned}
\text{spec } \hat{H} &= z^{np/2} \sum_{\eta} (-1)^{c_\eta} \binom{n}{\eta'} z^{|\eta|} \sum_{k \geqslant 0} \mu(E_k) z^k \\
&= z^{np/2} \sum_{k,\eta} (-1)^{c_\eta} \binom{n}{\eta'} \mu(E_k) z^{k+|\eta|} \\
&= z^{np/2} \sum_{l \geqslant 0} \sum_{k,\eta,\, k+|\eta|=l} (-1)^{c_\eta} \binom{n}{\eta'} \mu(E_k) z^l.
\end{aligned}$$

Here, the partitions $\eta$ are those of the form (3.6) and $c_\eta$ is given by (3.7). On the other hand, we clearly have from (3.3) that the spectrum generating function is also given by

$$\text{spec } \hat{H} = z^{np/2} \sum_{l \geqslant 0} \mu\big(E_l^{(p)}\big) z^l,$$

so that in fact

$$\mu\big(E_l^{(p)}\big) = \sum_{k,\eta,\, k+|\eta|=l} (-1)^{c_\eta} \binom{n}{\eta'} \mu(E_k), \qquad l = 0, 1, 2, \ldots \tag{5.17}$$

with the convention that $\mu(E_k) = 0$ for $k < 0$. The partitions $\eta$ are still of the form (3.6). Formula (5.17) thus gives in fact a way of determining the multiplicities $\mu\big(E_l^{(p)}\big)$ in terms of the 'generic' multiplicities $\mu(E_k)$ with $k \leqslant l$.

*Example: the case* $p = n-2$. We start by noting that $\binom{n}{\lambda} = 0$ for any partition where $\lambda_1 > n$. If we have a partition $\eta$ of the form (3.6), then $\eta'_1 = 1 + a_1 + (n-2)$, meaning that $\binom{n}{\eta'}$ has a chance of being non-zero only when $a_1 \leqslant 1$. It is now easy to enumerate all possible partitions $\eta$:

| | | | | |
|---|---|---|---|---|
| $\eta = (1, 0 \vert n-1, n-2)$ | $c_\eta = 1+2 = 3$ | $\vert\eta\vert = 2n$ | $\eta = (2^n)$ | $\binom{n}{\eta'} = 1$ |
| $\eta = (1 \vert n-1)$ | $c_\eta = 1+1 = 2$ | $\vert\eta\vert = n+1$ | $\eta = (2, 1^{n-1})$ | $\binom{n}{\eta'} = n$ |
| $\eta = (0 \vert n-2)$ | $c_\eta = 0+1 = 1$ | $\vert\eta\vert = n-1$ | $\eta = (1^{n-1})$ | $\binom{n}{\eta'} = n$ |
| $\eta = ()$ | $c_\eta = 0$ | $\vert\eta\vert = 0$ | $\eta = ()$ | $\binom{n}{\eta'} = 1$ |

.

We thus have that

$$\mu\big(E_l^{(n-2)}\big) = \mu(E_l) - n\mu(E_{l-n+1}) + n\mu(E_{l-n-1}) - \mu(E_{l-2n}), \tag{5.18}$$

again with the convention that $\mu(E_k) = 0$ for $k < 0$.

*The three-dimensional isotropic oscillator.* In the case when $n = 3$, there are only three cases to consider for the $\mathfrak{osp}(1|6)$ solution: $p > 2$, $p = 2$ or $p = 1$. For the generic case $p > 2$, the generating function for the multiplicities is simply

$$\frac{1}{(1-z)^3(1-z^2)^3} = \sum_{k\geqslant 0} \mu(E_k) z^k,$$

and the online encyclopedia of integer sequences ([33], A038163) then gives a closed form for the multiplicities:

$$\mu(E_{2k}) = \frac{4k+5}{5}\binom{k+4}{4} \qquad \text{and} \qquad \mu(E_{2k+1}) = \frac{4k+15}{5}\binom{k+4}{4}.$$

Note that these closed form expressions could also be obtained by putting $n = 3$ in the right-hand side of (5.14), and performing a generalization of Kummer's identity for hypergeometric series [34]. When $p = 2$, we can use (5.12) or (5.13), yielding

$$\mu\big(E_{2k}^{(2)}\big) = \frac{(k+1)^2(k+2)^2}{4}, \qquad \mu\big(E_{2k+1}^{(2)}\big) = \frac{(k+1)(k+2)^2(k+3)}{4},$$

or one can use (5.16), which after manipulation of the binomial coefficients yields the same explicit result. Finally, for $p = 1$, one has

$$\mu\big(E_k^{(1)}\big) = \binom{k+2}{k} = \frac{(k+2)(k+1)}{2}.$$

Numerically, this gives the following:

| $p$ | $\mu(E_0^{(p)})$ | $\mu(E_1^{(p)})$ | $\mu(E_2^{(p)})$ | $\mu(E_3^{(p)})$ | $\mu(E_4^{(p)})$ | $\mu(E_5^{(p)})$ | $\mu(E_6^{(p)})$ | $\mu(E_7^{(p)})$ | $\mu(E_8^{(p)})$ | $\mu(E_9^{(p)})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 |
| 2 | 1 | 3 | 9 | 18 | 36 | 60 | 100 | 150 | 225 | 315 |
| $p > 2$ | 1 | 3 | 9 | 19 | 39 | 69 | 119 | 189 | 294 | 434 |

Note how the multiplicity of the first two energy levels is unaffected by considering non-canonical solutions. Also, from this table one can check (5.18) numerically for some values.

## 6. Spectrum of the Hamiltonian in the $\mathfrak{gl}(1|n)$ solution

As already mentioned before, on an algebraic level, the Hamiltonian is given by

$$\hat{H} = \hbar\left(\beta e_{00} + \sum_{k=1}^{n} \beta_k e_{kk}\right),$$

with $\beta = \sum_{k=1}^{n} \beta_k$ and $\beta_k = -\omega_k + \frac{1}{n-1}\sum_{l=1}^{n} \omega_l$. When the oscillator is isotropic, however, all $\beta_k$'s are equal and given by

$$\beta_k = \frac{\omega}{n-1};$$

consequently

$$\beta = \frac{n\omega}{n-1}.$$

The representations of relevance are the unitary irreducible $\mathfrak{gl}(1|n)$ representations. We shall consider here only one class of such representations, namely the covariant representations $V_\lambda$ which are labelled by a partition $\lambda$ satisfying $\lambda_2 \leqslant n$. The character of such a representation $V_\lambda$ is given by the supersymmetric Schur function $s_\lambda(x_1|y_1,\ldots,y_n)$. So the first set of variables consists of one variable only, while the second set consists of $n$ variables. Formula (3.14) reduces in this case to

$$s_\lambda(x_1|y_1,\ldots,y_n) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_1) s_{\nu'}(y_1,\ldots,y_n), \tag{6.1}$$

while the Berele–Regev formula for partitions satisfying $\lambda_1 \geqslant n$ becomes

$$\begin{aligned} s_\lambda(x_1|y_1,\ldots,y_n) &= s_{(\lambda_1-n)}(x_1) s_{(\lambda_2,\lambda_3,\ldots)'}(y_1,\ldots,y_n) \prod_{j=1}^n (x_1+y_j) \\ &= x_1^{\lambda_1-n} s_{(\lambda_2,\lambda_3,\ldots)'}(y_1,\ldots,y_n) \prod_{j=1}^n (x_1+y_j). \end{aligned} \tag{6.2}$$

*The case $\lambda_1 \geqslant n$ (typical case).* Formula (6.2) allows us to draw conclusions about the spectrum of the Hamiltonian very easily. So, we start by concentrating on the case when the partition $\lambda$ is such that $\lambda_1 \geqslant n$. In order to determine the spectrum generating function, we have to perform the substitutions

$$x_1 \to t^{\hbar\beta} = t^{\frac{\hbar\omega n}{n-1}} =: z^n, \qquad y_j \to t^{\hbar\beta_j} = t^{\frac{\hbar\omega}{n-1}} =: z \qquad (j=1,\ldots,n). \tag{6.3}$$

With these substitutions and using (5.6), the spectrum generating function becomes

$$\begin{aligned} \operatorname{spec} \hat{H} &= z^{n(\lambda_1-n)} z^{|\lambda|-\lambda_1} \binom{n}{(\lambda_2,\lambda_3,\ldots)} (z^n+z)^n \\ &= z^{(n-1)(\lambda_1-n)+|\lambda|}(1+z^{n-1})^n \binom{n}{(\lambda_2,\lambda_3,\ldots)} \\ &= t^{\hbar\omega(|\lambda|/(n-1)+\lambda_1-n)}(1+t^{\hbar\omega})^n \binom{n}{(\lambda_2,\lambda_3,\ldots)}. \end{aligned} \tag{6.4}$$

From this, it is immediately clear that the ground energy level is given by

$$E_0^{(\lambda)} = \hbar\omega\left(\frac{|\lambda|}{n-1} + \lambda_1 - n\right). \tag{6.5}$$

Furthermore, we see that there are in fact $n+1$ different energy levels, equidistant with spacing $\hbar\omega$, and hence the highest energy level is

$$E_n^{(\lambda)} = \hbar\omega\left(\frac{|\lambda|}{n-1} + \lambda_1\right).$$

Also, from the binomial theorem, it is immediately clear that the multiplicity $\mu\big(E_k^{(\lambda)}\big)$ of the $k$th energy level is given by

$$\mu(E_k^{(\lambda)}) = \binom{n}{k}\binom{n}{(\lambda_2,\lambda_3,\ldots)},$$

which is the product of an ordinary and a generalized binomial coefficient. In [21], it was already shown that the multiplicity of the different energy levels in the representation $V_{(p)}$,

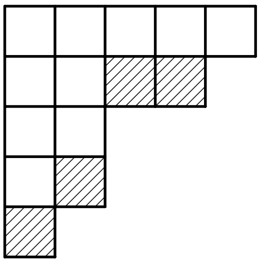


**Figure 4.** An example of a horizontal 4-strip. The partition $\lambda = (5, 4, 2, 2, 1)$ and $\nu = (5, 2, 2, 1)$.

with $p \geqslant n$, was in fact given by the binomial coefficients $\binom{n}{k}$, a fact recovered here, since the generalized binomial coefficient involves the empty partition. Note also that the multiplicity of the different energy levels does not depend on $\lambda_1$ (as long as $\lambda_1 \geqslant n$), but that $\lambda_1$ does influence the (height of the) ground level.

*The general case.* When $\lambda_1 < n$, no nice factorization of the supersymmetric Schur function $s_\lambda(x_1|y_1, \ldots, y_n)$ exists (the representation is then atypical), and hence we resort to using the expansion (6.1), which is valid in general. Since the set of variables $x$ is simply $x_1$, this means that in (6.1) the Schur function $s_\mu(x_1)$ vanishes unless $\mu = (r)$ for some non-negative integer $r$. More in particular, we then have

$$s_{(r)}(x_1) = h_r(x_1) = x_1^r.$$

Also the Littlewood–Richardson coefficients are particularly easy in this case, see ([24], section 5):

$$c^\lambda_{(r)\nu} = \begin{cases} 1 & \text{if} \quad \lambda - \nu \text{ is a horizontal } r\text{-strip} \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

Let $\lambda$ and $\nu$ be two partitions such that the Young diagram of $\nu$ is embedded in the Young diagram of $\lambda$, or stated otherwise $\nu_i \leqslant \lambda_i$, for all $i$. The set theoretic difference $\theta = \lambda - \nu$ is called a skew diagram. The skew diagram is called a horizontal strip if all $\theta'_i \leqslant 1$, or stated otherwise, if the diagram of $\theta$ contains at most one block per column. Naturally, a horizontal strip is a horizontal $r$-strip if it consists of exactly $r$ blocks. Figure 4 gives an example.

The Littlewood–Richardson coefficients given in (6.6) are in fact equivalent with Pieri's rule [24]:

$$s_{(r)}(x)s_\nu(x) = \sum_\lambda s_\lambda(x), \qquad \text{(valid for general } x\text{)}$$

where the sum is over all partitions $\lambda$ such that $\lambda - \nu$ is a horizontal $r$-strip. The expansion (6.1) may hence be written as

$$s_\lambda(x_1|y) = \sum_{r\geqslant 0} x_1^r \sum_\nu s_{\nu'}(y), \tag{6.7}$$

where the sum is over all partitions $\nu$ such that $\lambda - \nu$ is a horizontal $r$-strip. Performing the substitutions (6.3) yields the following spectrum generating function:

$$\text{spec } \hat{H} = \sum_{r\geqslant 0} t^{\frac{\hbar\omega n r}{n-1}} \sum_\nu t^{\frac{\hbar\omega|\nu|}{n-1}} \binom{n}{\nu} = \sum_{r\geqslant 0} t^{\hbar\omega(\frac{|\lambda|}{n-1}+r)} \sum_\nu \binom{n}{\nu}, \tag{6.8}$$

where the inner sum runs over all partitions $\nu$ such that $\lambda - \nu$ is a horizontal $r$-strip (and hence $|\nu| = |\lambda| - r$).

This spectrum generating function allows us to determine the highest energy level. Indeed, if $\theta = \lambda - \nu$ is a horizontal strip, then clearly $|\theta| \leqslant \lambda_1$, since $\theta$ is contained in $\lambda$. For each partition $\lambda$, there exists exactly one partition $\nu$ such that $\lambda - \nu$ is a horizontal $\lambda_1$-strip, namely the partition $\nu$ for which $\nu'_j = \lambda'_j - 1$ $(j = 1, \ldots, \lambda_1)$. For this partition one will have that $\binom{n}{\nu} > 0$, which is the same as claiming that $\nu_1 \leqslant n$. This follows from the fact that $\nu_1 = \lambda_2 \leqslant n$. The highest energy level is thus given by

$$\hbar\omega\left(\frac{|\lambda|}{n-1} + \lambda_1\right).$$

From (6.8) one would be tempted to conclude that the ground level corresponds to the level implied by $r = 0$. This is incorrect, however, since it may happen that the summation over $\nu$ vanishes. Remembering that the generalized binomial coefficient $\binom{n}{\lambda'}$ is in fact a specialization of the symmetric Schur function $s_\lambda$, see (5.6), it is clear that $\binom{n}{\lambda}$ is a non-negative integer whenever $n$ is. This means that

$$\sum_\nu \binom{n}{\nu} = 0 \iff \forall \nu : \binom{n}{\nu} = 0,$$

or stated otherwise, the summation vanishes if and only if each term vanishes. It is also clear that the non-negative integer roots of a generalized binomial coefficient $\binom{x}{\lambda}$ are exactly $\{0, 1, \ldots, \lambda_1 - 1\}$ (see figure 3). So, if for a particular $r$, all partitions $\nu$ for which $\lambda - \nu$ is a horizontal $r$-strip are such that $\nu_1 > n$, then the inner summation in (6.8) will vanish. The ground energy level is thus given by $\hbar\omega(|\lambda|/(n-1) + r^*)$, with

$$r^* = \min\{r | \exists \nu : \lambda - \nu \text{ is a horizontal } r\text{-strip and } \nu_1 \leqslant n\}.$$

Note that we have

$$r^* = \begin{cases} 0 & \text{if} \quad \lambda_1 \leqslant n \\ \lambda_1 - n & \text{otherwise.} \end{cases}$$

Indeed, if $\lambda_1 \leqslant n$, then taking $\nu = \lambda$ yields the horizontal 0-strip with $\nu_1 \leqslant n$. On the other hand, if $\lambda_1 > n$, then taking $\nu_1 = n$ and $\nu_j = \lambda_j$ for $j \geqslant 2$ yields a horizontal $(\lambda_1 - n)$-strip with $\nu_1 \leqslant n$. This is in agreement with what was found using the Berele–Regev formula, see (6.5).

It is also quite clear that all energy levels between the ground level and the most excited level do in fact exist, and hence we are dealing with an equidistant spectrum with spacing $\hbar\omega$ (just as in the $\mathfrak{osp}(1|2n)$ case). The number of different energy levels is now also easily determined since it is given by

$$\lambda_1 - r^* + 1 = \min\{\lambda_1, n\} + 1. \tag{6.9}$$

Writing the spectrum generating function in the following way:

$$\text{spec}\,\hat{H} = t^{\hbar\omega(\frac{|\lambda|}{n-1} + r^*)} \sum_{k=0}^{\min\{\lambda_1, n\}} \mu\big(E_k^{(\lambda)}\big) t^{\hbar\omega k},$$

it is clear that

$$\mu\big(E_k^{(\lambda)}\big) = \sum_\nu \binom{n}{\nu},$$

where the sum is over all partitions $\nu$ such that $\lambda - \nu$ is a horizontal $(r^* + k)$-strip. One striking difference with the $\mathfrak{osp}(1|2n)$ solutions is that now the multiplicity of the ground level, $\mu\big(E_0^{(\lambda)}\big)$, is not 1. A similarity is that $\mu\big(E_k^{(\lambda)}\big)$ is a polynomial in the variable $n$. The last statement is true for $n$ sufficiently large: in other words, one should think of $\lambda$ as being fixed, and $n$ increasing. When $n \geqslant \lambda_1, r^* = 0$, and then $\mu\big(E_k^{(\lambda)}\big)$ is a polynomial of degree $|\nu| = |\lambda| - k$ in the variable $n$.

As an application, we consider the spectrum of the Hamiltonian in the representation $V_{(1^p)}$. According to (6.9), there will be only two different energy levels. For the multiplicities we have

$$\mu\big(E_0^{(1^p)}\big) = \sum_\nu \binom{n}{\nu} = \binom{n}{(1^p)} = (-1)^p \binom{-n}{p} = \binom{n+p-1}{p}$$

and

$$\mu\big(E_1^{(1^p)}\big) = \sum_\nu \binom{n}{\nu} = \binom{n}{(1^{p-1})} = (-1)^{p-1} \binom{-n}{p-1} = \binom{n+p-2}{p-1}.$$

This is in agreement with what was found in [21], but there explicit knowledge of the representation actions was used, whereas here we have only used the character of the representation.

## 7. Conclusion

In this paper we have given a general method for determining the spectrum generating functions associated with an element of the Cartan subalgebra of a Lie superalgebra $\mathfrak{g}$. These spectrum generating functions are closely related to the character of the studied representations.

This method was used to study the energy spectrum of the $n$-dimensional isotropic harmonic oscillator when viewed as a WQS. Two different solutions of the compatibility conditions of this system were considered, one being related to the $\mathfrak{osp}(1|2n)$ Lie superalgebra and the other to the $\mathfrak{gl}(1|n)$ Lie superalgebra. The spectrum generating functions involved interesting specializations of Schur symmetric and supersymmetric functions. In both cases the energy spectrum is equally spaced with spacing $\hbar\omega$. In the $\mathfrak{osp}(1|2n)$ case, we considered the representations $V(p)$ and the energy spectrum is then countably infinite, and degeneracies when $p \in \{1, 2, \ldots, n-1\}$ (non-generic situation) may be determined in function of the degeneracies in the case $p > n-1$ (generic situation). In the $\mathfrak{gl}(1|n)$ case, since we considered the finite dimensional representations $V_\lambda$, the spectrum is of course finite. Also here, one can speak of a generic case $\lambda_1 \geqslant n$ and non-generic cases $\lambda_1 < n$. In the first case, one may use the Berelev–Regev formula to easily determine the energy spectrum and associated multiplicities.

It is, however, also possible to apply the method to study the spectrum of a linear chain involving quadratic nearest neighbour interactions. Although one probably will end up with more involved specializations of the Schur functions, it should still be possible to deduce general findings about the energy spectrum. In particular, when considering chains of harmonic oscillators as in [20], one could try to find out for which values of the coupling constant $c$ the spurious degeneracies occur. This could be part of future work.

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